

# STEKLOV APPROXIMATIONS OF HARMONIC BOUNDARY VALUE PROBLEMS ON PLANAR REGIONS.

GILES AUCHMUTY AND MANKI CHO

**ABSTRACT.** Error estimates for approximations of harmonic functions on planar regions by subspaces spanned by the first harmonic Steklov eigenfunctions are found. They are based on the explicit representation of harmonic functions in terms of these harmonic Steklov eigenfunctions. When the region is a rectangle of aspect ratio  $h$ , some computational results regarding these approximations for problems with known explicit solutions are described.

## 1. INTRODUCTION

This paper will describe results about the approximation of solutions of various boundary value problems for Laplace's equations on a bounded planar regions  $\Omega$  in the plane using certain harmonic Steklov eigenfunctions. That is our interest is in approximating harmonic functions  $u : \overline{\Omega} \rightarrow \mathbb{R}$  that satisfy either Dirichlet, Robin or Neumann boundary conditions

$$u = g \quad \text{or} \quad D_\nu u + b u = g \quad \text{on} \quad \partial\Omega \quad (1.1)$$

Here  $\nu$  is the outward unit normal and  $b \geq 0$  is a constant.

It has been shown that there are orthogonal bases of the class of all finite energy harmonic functions  $\mathcal{H}(\Omega)$  on  $\Omega$  consisting of harmonic Steklov eigenfunctions - as summarized below in section 3. A natural question is how good are approximations using specific subclasses of these eigenfunctions; particularly those corresponding to the lowest Steklov eigenvalues? Here some general results about these approximations are described in sections 4 and 6 and some computational results for particular problems with exact solutions are described in sections 5 and 7. The computational results are based on the fact that explicit formulae are known for the Steklov eigenvalues and eigenfunctions on rectangles of arbitrary aspect ratio  $h$ .

Existence-uniqueness theorems for these problems may be found in most texts that treat elliptic boundary value problems. Under differing assumptions on  $g$ ,  $\Omega$  and  $\partial\Omega$  the solutions  $u$  are  $C^\infty$  on  $\Omega$  and lie in various different Banach or Hilbert spaces of functions on  $\Omega$ . For an excellent review of classical results about these problems see chapter 2 of

---

*Date:* September 26, 2016; SAHFin.tex.

The first author gratefully acknowledges research support by NSF award DMS 11008754.

*2010 Mathematics Subject classification.* Primary 65M70, Secondary 65N25, 31B05.

*Key words and phrases.* Harmonic functions, Steklov eigenfunctions, boundary value problems, harmonic approximation .

[12] by Benilan. In particular a function  $u \in L^1(\Omega)$  is said to be an *ultraweak* solution of Laplace's equation provided it obeys

$$\int_{\Omega} u \Delta \varphi \, dx dy = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \quad (1.2)$$

Such an ultraweak solution is a *classical solution* of Laplace's equation provided it is equivalent to a continuous function on  $\bar{\Omega}$ . Such solutions need not be weak solutions in the Sobolev space  $H^1(\Omega)$  - even when  $\Omega$  is a disk in the plane.

In the sections 3, 4 and 6, general results about Steklov approximations of harmonic functions will be described. These are based on the observation that there is an algorithm for constructing a basis of classes of harmonic functions on regions consisting of harmonic Steklov eigenfunctions. See Auchmuty [1]-[3] and the boundary traces of these eigenfunctions are bases of associated Hilbert spaces of functions on  $\partial\Omega$ . In particular various error estimates for Steklov approximations are obtained.

When the domain is a planar disk, the Steklov eigenfunctions are the usual harmonic functions  $r^m \cos m\theta, r^m \sin m\theta$  of Fourier analysis and the question of the approximation of harmonic functions on the unit disc by harmonic polynomials has a huge literature. The text of Axler, Bourdon and Ramey [7] is a recent introduction to the theory.

The use of Steklov bases for spaces of harmonic functions permits the generalization of some the results of classical harmonic function theory to quite general bounded (and even exterior) regions in  $\mathbb{R}^N$ , see [3]. Here some results about the approximation of harmonic functions by finite sums of Steklov eigenfunctions will be described with computational results for the case of a rectangle.

When the region  $\Omega$  is a rectangle, the Steklov eigenfunctions are known explicitly see Auchmuty and Cho [6] or Girouard and Polterovich [14] where a completeness proof may be found. The paper [6] described the generalization of the mean value theorem to rectangles and to cases where Robin data on  $\partial\Omega$  is known. Here in sections 5 and 7 computational results for Steklov approximations of certain harmonic functions regarded as solutions of Laplace's equations with various boundary value conditions are described.

For general regions, the Steklov eigenvalues and eigenfunctions are not (yet) known explicitly. However a number of authors have studied the numerical determination of these eigenfunctions including Cheng, Lin and Zhang [8], and Kloucek, Sorensen and Wightman [17]. The software FreeFem++ [16] has subroutines for the computation of Steklov eigenfunctions and eigenvalues that was used for confirmation of some of the analytical results described here.

After introducing our assumptions and notation in section 2, the basic properties of harmonic Steklov eigenproblems are described in section 3. In particular the formula for the  $(L^2-)$ harmonic extension operator - or solution operator for the Dirichlet harmonic problem is described. In section 4 some error estimates for Steklov approximations are given. Then some computational results about such problems on a rectangle are described in section 5. Dirichlet problems are considered in sections 4 and 5 and then results for Robin and Neumann problems are described in sections 6 and 7.

Our general conclusion is that the use of Steklov approximations of harmonic functions should be seriously considered by researchers needing good approximations using a small dimensional subspace of harmonic functions. Once the Steklov eigenfunctions and eigenvalues of a region are known, simple formulae provide excellent approximations of the solutions in a region. Properties of the approximations close to, or on, the boundary needs significant further study. It should be noted that this analysis extends to the solution much more general self-adjoint second order elliptic equations of the form  $\mathcal{L}u = 0$  using similar general constructions.

## 2. ASSUMPTIONS AND NOTATION.

This paper treats various Laplacian boundary value problems on regions  $\Omega$  in the plane  $\mathbb{R}^2$ . A region is a non-empty, connected, open subset of  $\mathbb{R}^2$ . Its closure is denoted  $\overline{\Omega}$  and its boundary is  $\partial\Omega := \overline{\Omega} \setminus \Omega$ . Some regularity of the boundary  $\partial\Omega$  is required. Each component (= maximal connected closed subset) of the boundary is assumed to be a Lipschitz continuous closed curve. Let  $\sigma$  denote arc-length along a curve so the unit outward normal  $\nu(z)$  is defined  $\sigma$  a.e.

$L^p(\Omega)$  and  $L^p(\partial\Omega, d\sigma)$ ,  $1 \leq p \leq \infty$  are the usual spaces with p-norm denoted by  $\|u\|_p$  or  $\|u\|_{p,\partial\Omega}$  respectively. When  $p = 2$  these are real Hilbert spaces with inner products defined by

$$\langle u, v \rangle := \int_{\Omega} u(x) v(x) dx \quad \text{and} \quad \langle u, v \rangle_{\partial\Omega} := |\partial\Omega|^{-1} \int_{\partial\Omega} u v d\sigma.$$

$C(\overline{\Omega})$  is the space of continuous functions on the closure  $\overline{\Omega}$  of  $\Omega$  with the sup norm  $\|u\|_b := \sup_{\overline{\Omega}} |u(x, y)|$ .

The weak j-th derivative of  $u$  is  $D_j u$  - and all derivatives will be taken in a weak sense. Then  $\nabla u(x) := (D_1 u(x), \dots, D_N u(x))$  is the gradient of  $u$  and  $H^1(\Omega)$  is the usual real Sobolev space of functions on  $\Omega$ . It is a real Hilbert space under the standard  $H^1$ -inner product

$$[u, v]_1 := \int_{\Omega} [u(x) \cdot v(x) + \nabla u(x) \cdot \nabla v(x)] dx. \quad (2.1)$$

The corresponding norm is denoted  $\|u\|_{1,2}$ .

The region  $\Omega$  is said to satisfy *Rellich's theorem* provided the imbedding of  $H^1(\Omega)$  into  $L^p(\Omega)$  is compact for  $1 \leq p < \infty$ .

The boundary trace operator  $\gamma : H^1(\Omega) \rightarrow L^2(\partial\Omega, d\sigma)$  is the linear extension of the map restricting Lipschitz continuous functions on  $\overline{\Omega}$  to  $\partial\Omega$ . The region  $\Omega$  is said to satisfy a *compact trace theorem* provided the boundary trace mapping  $\gamma : H^1(\Omega) \rightarrow L^2(\partial\Omega, d\sigma)$  is compact. Usually  $\gamma$  is omitted so  $u$  is used in place of  $\gamma(u)$  for the trace of a function on  $\partial\Omega$ .

The *Gauss-Green* theorem holds on  $\Omega$  provided

$$\int_{\Omega} u(x) D_j v(x) dx = \int_{\partial\Omega} \gamma(u) \gamma(v) \nu_j d\sigma - \int_{\Omega} v(x) D_j u(x) dx \quad \text{for } 1 \leq j \leq N. \quad (2.2)$$

for all  $u, v$  in  $H^1(\Omega)$ . The requirements on the region will be

**Condition B1:**  $\Omega$  is a bounded region in  $\mathbb{R}^2$  whose boundary  $\partial\Omega$  is a finite number of disjoint closed Lipschitz curves, each of finite length and such that the Gauss-Green, Rellich and compact trace theorems hold.

We will use the equivalent inner products on  $H^1(\Omega)$  defined by

$$[u, v]_{\partial} := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u v \, d\sigma. \quad (2.3)$$

The corresponding norm will be denoted by  $\|u\|_{\partial}$ . The proof that this norm is equivalent to the usual  $(1, 2)$ -norm on  $H^1(\Omega)$  when (B1) holds is Corollary 6.2 of [1] and also is part of theorem 21A of [19].

A function  $u \in C(\overline{\Omega})$  or  $H^1(\Omega)$  is said to be harmonic provided it satisfies (1.2). Define  $\mathcal{H}(\Omega)$  to be the space of all harmonic functions in  $H^1(\Omega)$ . When (B1) holds, the closure of  $C_c^1(\Omega)$  in the  $H^1$ -norm is the usual Sobolev space  $H_0^1(\Omega)$ . Then (1.2) is equivalent to saying that  $\mathcal{H}(\Omega)$  is  $\partial$ -orthogonal to  $H_0^1(\Omega)$ . This may be expressed as

$$H^1(\Omega) = H_0^1(\Omega) \oplus_{\partial} \mathcal{H}(\Omega), \quad (2.4)$$

where  $\oplus_{\partial}$  indicates that this is a  $\partial$ -orthogonal decomposition.

The analysis to be described here is based on the construction of a  $\partial$ -orthogonal basis of the Hilbert space  $\mathcal{H}(\Omega)$  consisting of harmonic Steklov eigenfunctions. In particular we shall prove results about the approximation of solutions of harmonic boundary value problems by such eigenfunctions.

### 3. STEKLOV REPRESENTATIONS OF SOLUTIONS OF HARMONIC BOUNDARY VALUE PROBLEMS.

Let  $\Omega$  be a bounded region in  $\mathbb{R}^2$  that satisfies (B1). A non-zero function  $s \in H^1(\Omega)$  is said to be a *harmonic Steklov eigenfunction* on  $\Omega$  corresponding to the Steklov eigenvalue  $\delta$  provided  $s$  satisfies

$$\int_{\Omega} \nabla s \cdot \nabla v \, dx = \delta \langle s, v \rangle_{\partial\Omega} = \delta |\partial\Omega|^{-1} \int_{\partial\Omega} s v \, d\sigma. \quad \text{for all } v \in H^1(\Omega). \quad (3.1)$$

This is the weak form of the boundary value problem

$$\Delta s = 0 \quad \text{on } \Omega \text{ with } D_{\nu} s = \delta |\partial\Omega|^{-1} s \quad \text{on } \partial\Omega. \quad (3.2)$$

Here  $\Delta$  is the Laplacian and  $D_{\nu} s(x) := \nabla s(x) \cdot \nu(x)$  is the unit outward normal derivative of  $s$  at a point on the boundary.

Descriptions of the analysis of these eigenproblems may be found in Auchmuty [1] - [4]. These eigenvalues and a corresponding family of  $\partial$ -orthonormal eigenfunctions may be found using variational principles as described in sections 6 and 7 of Auchmuty [1].  $\delta_0 = 0$  is the least eigenvalue of this problem corresponding to the eigenfunction  $s_0(x) \equiv 1$  on  $\Omega$ . This eigenvalue is simple as  $\Omega$  is connected. Let the first  $k$  Steklov eigenvalues be  $0 = \delta_0 < \delta_1 \leq \delta_2 \leq \dots \leq \delta_{k-1}$  and  $s_0, s_1, \dots, s_{k-1}$  be a corresponding set

of  $\partial$ -orthonormal eigenfunctions. The  $k$ -th eigenfunction  $s_k$  will be a maximizer of the functional

$$\mathcal{B}(u) := \int_{\partial\Omega} |\gamma(u)|^2 d\sigma \quad (3.3)$$

over the subset  $B_k$  of functions in  $H^1(\Omega)$  which satisfy

$$\|u\|_{\partial} \leq 1 \quad \text{and} \quad \langle \gamma(u), \gamma(s_l) \rangle_{\partial\Omega} = 0 \quad \text{for} \quad 0 \leq l \leq k-1. \quad (3.4)$$

The existence and some properties of such eigenfunctions are described in sections 6 and 7 of [1] for a more general system. In particular, that analysis shows that each  $\delta_j$  is of finite multiplicity and  $\delta_j \rightarrow \infty$  as  $j \rightarrow \infty$ ; see Theorem 7.2 of [1]. The maximizers not only are  $\partial$ -orthonormal but they also satisfy

$$\int_{\Omega} \nabla s_k \cdot \nabla s_l dx = |\partial\Omega|^{-1} \int_{\partial\Omega} s_k s_l d\sigma = 0 \quad \text{for} \quad k \neq l. \quad (3.5)$$

$$\int_{\Omega} |\nabla s_k|^2 dx = \frac{\delta_k}{1 + \delta_k} \quad \text{and} \quad |\partial\Omega|^{-1} \int_{\partial\Omega} |\gamma(s_k)|^2 d\sigma = \frac{1}{1 + \delta_k} \quad \text{for} \quad k \geq 0. \quad (3.6)$$

Recently Daners [11] corollary 4.3 has shown that, when  $\Omega$  is a Lipschitz domain, then the Steklov eigenfunctions are continuous on  $\overline{\Omega}$ .

The analysis in this paper is based on the fact that harmonic Steklov eigenfunctions on  $\Omega$  can be chosen to be orthogonal bases of both  $\mathcal{H}(\Omega)$  and of  $L^2(\partial\Omega, d\sigma)$ . It should be noted that, for regions other than discs (or balls in higher dimensions), these Steklov eigenfunctions are generally not  $L^2$ -orthogonal on  $\Omega$ .

Let  $\mathcal{S} := \{s_j : j \geq 0\}$  be the maximal family of  $\partial$ -orthonormal eigenfunctions constructed inductively as above. For this paper, it is more convenient to use the Steklov eigenfunctions normalized by their boundary norms.

Define the functions  $\tilde{s}_j(x) := \sqrt{1 + \delta_j} s_j(x)$  for  $j \geq 0$ . From (3.6), these satisfy

$$\int_{\partial\Omega} \tilde{s}_j \tilde{s}_k d\sigma = 0 \quad \text{when} \quad j \neq k \quad \text{and} \quad \int_{\partial\Omega} \tilde{s}_j^2 d\sigma = |\partial\Omega|. \quad (3.7)$$

These Steklov eigenfunctions are said to be *boundary normalized* and the associated set  $\tilde{\mathcal{S}} := \{\tilde{s}_j : j \geq 0\}$  is an orthonormal basis of  $L^2(\partial\Omega, d\sigma)$ . See theorem 4.1 of [2].

For given  $g \in L^2(\partial\Omega, d\sigma)$ , let

$$g_M(x, y) := \bar{g} + \sum_{j=1}^M \hat{g}_j \tilde{s}_j(x, y) \quad \text{with} \quad \hat{g}_j = \langle g, \tilde{s}_j \rangle_{\partial\Omega} \quad (3.8)$$

be the  $M$ -th Steklov approximation of  $g$  on  $\partial\Omega$ . Here  $\bar{g} := g_0$  is the mean value of  $g$  on  $\partial\Omega$  and this is the standard projection of elements in a Hilbert space onto subsets of an orthonormal basis. Note that  $g_M$  is continuous and bounded on  $\partial\Omega$  as each  $\tilde{s}_j$  is and  $g_M$  converges strongly to  $g$  in  $L^2(\partial\Omega, d\sigma)$  from the Riesz-Fischer theorem and

$$\|g - g_M\|_{2, \partial\Omega}^2 = \|g\|_{2, \partial\Omega}^2 - \|g_M\|_{2, \partial\Omega}^2. \quad (3.9)$$

The unique solution of Laplace's equation on  $\Omega$  subject to the Dirichlet boundary condition  $\gamma(u) = g$  on  $\partial\Omega$  is given by

$$u(x, y) = E_H g(x, y) = \bar{g} + \lim_{M \rightarrow \infty} \sum_{j=1}^M \hat{g}_j \tilde{s}_j(x, y) \quad \text{for } (x, y) \in \Omega. \quad (3.10)$$

See section 6 of [3] for a proof.  $E_H$  will be called the *harmonic extension* operator and is a compact linear map from  $L^2(\partial\Omega, d\sigma)$  to  $L^2(\Omega)$ . Classically this map has been represented as an integral operator with the *Poisson kernel*. Theorem 6.3 of [3] says that  $E_H$  is an isometric isomorphism of  $L^2(\partial\Omega, d\sigma)$  with a space denoted  $\mathcal{H}^{1/2}(\Omega)$  that is a proper subspace of  $L^2(\Omega)$ .

#### 4. ERROR ESTIMATES FOR STEKLOV APPROXIMATIONS

First some error estimates for the Steklov approximations of solutions of the Dirichlet problem for Laplace's equation may be described. Essentially bounds on the error from these formulae for Steklov approximations are found that depend only on the errors in the Steklov approximations of  $g$  on the boundary. Let

$$u_M(x, y) := \bar{u} + \sum_{j=1}^M c_j s_j(x, y) \quad \text{for } (x, y) \in \Omega. \quad (4.1)$$

be a finite sum of the first  $M+1$  harmonic Steklov eigenfunctions on  $\Omega$ . Then for each integer  $M$ ,  $u_M$  is  $C^\infty$  on  $\Omega$ , continuous on  $\bar{\Omega}$  and in  $\mathcal{H}(\Omega)$ . The subspace spanned by  $\mathcal{S}_M := \{s_j : 0 \leq j \leq M\}$  will be denoted  $V_M$ .

**Theorem 4.1.** *Assume  $\Omega, \partial\Omega$  satisfy (B1) and  $\mathcal{S}, \tilde{\mathcal{S}}$  are the orthonormal bases of  $\mathcal{H}(\Omega)$ ,  $L^2(\partial\Omega, d\sigma)$  described above. If  $g \in C(\partial\Omega)$ ,  $u = E_H g$  and  $u_M$  is defined by (4.1), then*

$$\|u - u_M\|_{2,\Omega} \leq C_2 \|g - g_M\|_{2,\partial\Omega} \quad \text{and} \quad \|u - u_M\|_{\infty,\Omega} \leq \|g - g_M\|_{\infty,\partial\Omega} \quad (4.2)$$

where  $C_2$  is the Fichera constant for  $\Omega$  and  $g_M = \gamma(u_M)$ .

*Proof.* Here  $g_M$  is the boundary trace of  $u_M$ , so  $u_M$  is the harmonic extension of  $g_M$ . The 2-norm inequality is Fichera's inequality with  $C_2$  being the first eigenvalue of the Dirichlet, biharmonic Steklov eigenproblem. See [13] for the original version and [5] for a recent description and proof under weak boundary regularity conditions. The second inequality is the maximum principle for classical solutions of Laplace's equation.  $\square$

It should be noted here that the inequalities in (4.2) do not require that  $g_M$  be the Steklov approximation of  $g$  on  $\partial\Omega$ . They hold for any function that is a linear combination of the first  $M+1$  Steklov eigenfunctions. Note that  $L^p$ -bounds follow for  $2 < p < \infty$  by interpolation. When  $g_M$  is the  $M$ -th Steklov approximation of  $g$  on  $\partial\Omega$  as in (3.8), then one also has

**Theorem 4.2.** *Assume (B1) and  $g \in H^{1/2}(\partial\Omega)$ ,  $g_M$  is defined by (3.8),  $u = E_H g$  and  $u_M = E_H g_M$ . Then  $g_M$  converges strongly to  $g$  in  $H^{1/2}(\partial\Omega)$  and  $u_M$  converges uniformly to  $u$  on compact subsets of  $\Omega$ . Moreover*

$$\|\nabla(u - u_M)\|_{2,\Omega}^2 = \sum_{j=M+1}^{\infty} \delta_j \hat{g}_j^2 = \|g\|_{1/2,\partial\Omega}^2 - \|g_M\|_{1/2,\partial\Omega}^2 \quad (4.3)$$

*Proof.* The fact that  $g_M$  converges strongly to  $g$  in  $H^{1/2}(\partial\Omega)$  and  $H^1(\Omega)$  follows from the fact that  $\mathcal{S}$  is an orthonormal basis of  $\mathcal{H}(\Omega)$ . The proof of uniform convergence is standard, while (4.3) follows from the orthogonality properties of Steklov eigenfunctions.  $\square$

Also note that the Steklov eigenfunction have scaling properties. Given  $\Omega_1 \subset \mathbb{R}^2$ , let  $\Omega_L := \{Lx : x \in \Omega_1\}$  with  $L > 0$ . When  $h$  is a harmonic function on  $\Omega_1$ , then the function  $h_L(y) := h(y/L)$  will be a harmonic function on  $\Omega_L$ . If  $h$  is a harmonic Steklov eigenfunction on  $\Omega_1$  with Steklov eigenvalue  $\delta$ , then  $h_L$  will be a harmonic Steklov eigenfunction on  $\Omega_L$  with the Steklov eigenvalue  $\delta/L$ . Thus it suffices to study problems with a normalized bounded region  $\Omega_1$ ; the eigenvalues and eigenfunctions for scalings of a region then follow from these formulae.

The following sections will look at some aspects of the approximation of solutions of Laplace's equation on rectangles by finite sums of the form (3.10). Rectangles are chosen since we have explicit expressions for the Steklov eigenfunctions and eigenvalues on rectangles.

## 5. STEKLOV APPROXIMATIONS OF HARMONIC FUNCTIONS ON A RECTANGLE

When  $\Omega = R_h := (-1, 1) \times (-h, h)$  is a rectangle with aspect ratio  $h$ , the Steklov eigenfunctions and eigenvalues are known explicitly. See Auchmuty and Cho [6] section 4 where eight families of eigenfunctions are described and characterized by their symmetry properties with respect to the center. Class I eigenfunctions are even in  $x$  and  $y$ , class II are odd in  $x$  and  $y$ , class III are even in  $x$  and odd in  $y$ , class IV are odd in  $x$  and even in  $y$ .

By separation of variables the explicit formulae for the Steklov eigenfunctions may be found. The first eigenfunction  $s_0(x, y) \equiv 1$  is in class I and the other (unnormalized) Steklov eigenfunctions have the forms

$$s(x, y) := \cosh \nu x \cos \nu y \quad \text{when} \quad \tan \nu h + \tanh \nu = 0, \quad (5.1)$$

$$s(x, y) := \cos \nu x \cosh \nu y \quad \text{when} \quad \tan \nu + \tanh \nu h = 0. \quad (5.2)$$

When  $h = 1$ , the first eigenfunction in class II is  $s_3(x, y) = xy$ . Otherwise the (unnormalized) eigenfunctions and eigenvalues in this class have the forms

$$s(x, y) := \sinh \nu x \sin \nu y \quad \text{when} \quad \cot \nu h - \coth \nu = 0, \quad (5.3)$$

$$s(x, y) := \sin \nu x \sinh \nu y \quad \text{when} \quad \cot \nu - \coth \nu h = 0. \quad (5.4)$$

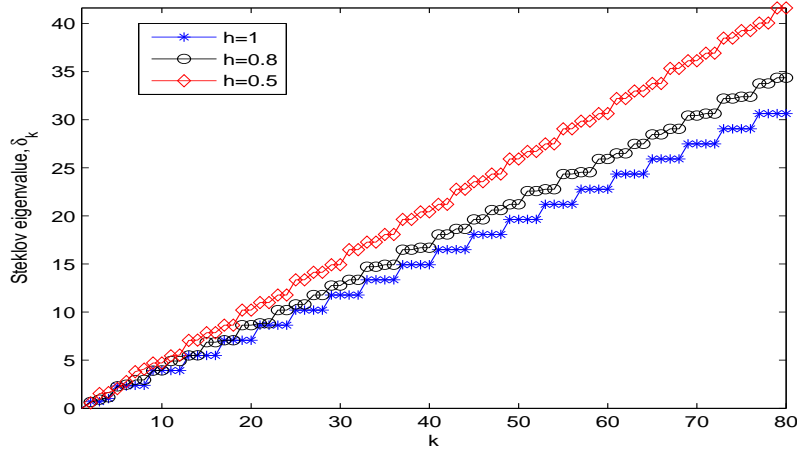


FIGURE 1. First 80 Steklov eigenvalues on  $R_h$  corresponding to  $h = 1, 0.8$ , and  $0.5$

Similarly eigenfunctions in class III have the forms

$$s(x, y) := \cosh \nu x \sin \nu y \quad \text{when} \quad \cot \nu h - \tanh \nu = 0, \quad (5.5)$$

$$s(x, y) := \cos \nu x \sinh \nu y \quad \text{when} \quad \tan \nu + \coth \nu h = 0 \quad (5.6)$$

Finally the eigenfunctions in class IV have the forms

$$s(x, y) := \sinh \nu x \cos \nu y \quad \text{when} \quad \tan \nu h + \coth \nu = 0 \quad (5.7)$$

$$s(x, y) := \sin \nu x \cosh \nu y \quad \text{when} \quad \cot \nu - \tanh \nu h = 0. \quad (5.8)$$

The associated Steklov eigenvalues,  $\delta$  are

- (i)  $\delta = \nu \tanh \nu$  when  $\nu$  is a solution of the equation in (5.1) or (5.5).
- (ii)  $\delta = \nu \tanh \nu h$  when  $\nu$  is a solution of the equation in (5.2) or (5.8).
- (iii)  $\delta = \nu \coth \nu$  when  $\nu$  is a solution of the equation in (5.3) or (5.7).
- (iv)  $\delta = \nu \coth \nu h$  when  $\nu$  is a solution of the equation in (5.4) or (5.6).

Knowing these explicit formulae for the eigenvalues and eigenfunctions the approximations of some given harmonic functions using relatively few harmonic Steklov eigenfunctions will be computed. Since there are eight families of harmonic Steklov eigenfunctions associated with different even/odd symmetries about the center we have concentrated on approximations involving the first  $8M$  eigenfunctions with  $M = 2, 3$  and  $5$ .

Note that the convergence results for the Steklov series expansions hold only when the coefficients are precisely the Steklov coefficients  $\hat{g}_j$  defined by (3.8). The value of  $\bar{u}$  is the mean value of the integral of  $g$  around  $\partial\Omega$ . However the approximation results of section 4 hold quite generally for any choice of coefficients.

For the following calculations the coefficients were obtained by evaluating the boundary integrals  $\hat{g}_j$  of (3.8) using the global adaptive quadrature (MATLAB's integral). The



absolute and relative error tolerance are  $10^{-10}$  and  $10^{-6}$ , respectively. Then the  $M$ -th Steklov approximation  $u_M$  is the function defined by (4.1) with  $\hat{u}_j = \hat{g}_j$ .

The following tables illustrate the pointwise approximations obtained for these sums at the points  $P_1 = (0.9, 0.9)$ ,  $P_2 = (0.9, 0.1)$ ,  $P_3 = (0.8, 0.6)$ ,  $P_4 = (0.3, 0.9)$ ,  $P_5 = (0.5, 0.5)$  and for  $M = 2, 3, 5$  and the exact results to 6 decimal places. Let  $D_M(x, y) := |g(x, y) - g_M(x, y)|$  be the absolute error at  $(x, y)$ . Also let  $f_1(x, y) := x^4 - 6x^2y^2 + y^4$ ,  $f_2(x, y) := \frac{2-x}{(2-x)^2+y^2}$ , and  $f_3(x, y) := \ln(\sqrt{(x-3)^2 + (y-3)^2})$ .

	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
M=2	-2.626748	0.694643	-0.844238	0.230283	-0.249859
M=3	-2.625942	0.607979	-0.842944	0.225907	-0.249983
M=5	-2.624712	0.607588	-0.843208	0.226837	-0.250000
$g(x, y)$	-2.624400	0.607600	-0.843200	0.226800	-0.250000
$D_2(x, y)$	0.002348	0.002957	0.001038	0.003483	0.000141
$D_3(x, y)$	0.001542	0.000379	0.000256	0.000893	0.000017
$D_5(x, y)$	0.000312	0.000012	0.000008	0.000037	0

TABLE 1.  $g(x, y) = f_1(x, y)$  and  $h = 1$ 

	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
M=2	0.544285	0.899505	0.666815	0.455438	0.600096
M=3	0.544745	0.902138	0.667202	0.460368	0.599985
M=5	0.544675	0.901609	0.666636	0.459219	0.600000
$g(x, y)$	0.544554	0.901639	0.666667	0.459459	0.600000
$D_2(x, y)$	0.000269	0.002135	0.000148	0.004021	0.000096
$D_3(x, y)$	0.000191	0.000498	0.000535	0.000909	0.000015
$D_5(x, y)$	0.000121	0.000030	0.000031	0.000240	0

TABLE 2.  $g(x, y) = f_2(x, y)$  and  $h = 1$ 

	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
M=2	1.088867	1.277069	1.179619	1.230746	1.262756
M=3	1.088349	1.274927	1.180394	1.229961	1.262881
M=5	1.088384	1.275412	1.180439	1.229874	1.262864
$g(x, y)$	1.088511	1.275503	1.180427	1.229794	1.262864
$D_2(x, y)$	0.000356	0.001566	0.000808	0.000952	0.000108
$D_3(x, y)$	0.000162	0.000576	0.000033	0.000167	0.000017
$D_5(x, y)$	0.000127	0.000091	0.000012	0.000080	0

TABLE 3.  $g(x, y) = f_3(x, y)$  and  $h = 1$ 

Let  $\text{rerr}_\infty(g) := \frac{\|g - g_M\|_{\infty, \partial\Omega}}{\|g\|_{\infty, \partial\Omega}}$  and  $\text{rerr}_2(g) := \frac{\|g - g_M\|_{2, \partial\Omega}}{\|g\|_{2, \partial\Omega}}$  be the relative error of  $M$ -th Steklov approximation of  $g$  in  $L^\infty(\Omega)$  norm and  $L^2(\partial\Omega, d\sigma)$ , respectively.

	$\text{rerr}_\infty(f_1)$	$\text{rerr}_\infty(f_2)$	$\text{rerr}_\infty(f_3)$
M=2	$6.59553 \times 10^{-3}$	$1.82382 \times 10^{-2}$	$6.48245 \times 10^{-3}$
M=3	$2.28748 \times 10^{-3}$	$1.21554 \times 10^{-2}$	$4.3219 \times 10^{-3}$
M=5	$5.55757 \times 10^{-4}$	$7.35222 \times 10^{-3}$	$2.59338 \times 10^{-3}$

TABLE 4. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 1$

	$\text{rerr}_\infty(f_1)$	$\text{rerr}_\infty(f_2)$	$\text{rerr}_\infty(f_3)$
M=2	$4.82556 \times 10^{-2}$	$2.46749 \times 10^{-2}$	$6.38229 \times 10^{-3}$
M=3	$4.20662 \times 10^{-2}$	$1.78505 \times 10^{-2}$	$4.18945 \times 10^{-3}$
M=5	$2.28023 \times 10^{-2}$	$1.0105 \times 10^{-2}$	$2.47618 \times 10^{-3}$

TABLE 5. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 0.8$

	$\text{rerr}_\infty(f_1)$	$\text{rerr}_\infty(f_2)$	$\text{rerr}_\infty(f_3)$
M=2	$2.09505 \times 10^{-1}$	$3.40908 \times 10^{-2}$	$5.58445 \times 10^{-3}$
M=3	$1.12233 \times 10^{-1}$	$2.00031 \times 10^{-2}$	$3.84456 \times 10^{-3}$
M=5	$7.66842 \times 10^{-2}$	$1.29479 \times 10^{-2}$	$2.24773 \times 10^{-3}$

TABLE 6. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 0.5$

	$\text{rerr}_2(f_1)$	$\text{rerr}_2(f_2)$	$\text{rerr}_2(f_3)$
M=2	$5.22051 \times 10^{-3}$	$1.30532 \times 10^{-2}$	$2.9694 \times 10^{-3}$
M=3	$1.57535 \times 10^{-3}$	$7.2083 \times 10^{-3}$	$1.62779 \times 10^{-3}$
M=5	$3.1167 \times 10^{-4}$	$3.43748 \times 10^{-3}$	$7.59478 \times 10^{-4}$

TABLE 7. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 1$

	$\text{rerr}_2(f_1)$	$\text{rerr}_2(f_2)$	$\text{rerr}_2(f_3)$
M=2	$5.13497 \times 10^{-2}$	$1.69181 \times 10^{-2}$	$2.77799 \times 10^{-3}$
M=3	$4.15782 \times 10^{-2}$	$1.0364 \times 10^{-2}$	$1.52184 \times 10^{-3}$
M=5	$1.78172 \times 10^{-2}$	$4.58322 \times 10^{-3}$	$6.98156 \times 10^{-4}$

TABLE 8. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 0.8$

It was observed that the above approximations were improved when some preliminary processing was performed. In particular it was worthwhile to first find the coefficients  $a_j$  for a function  $g_0(x, y) = a_0 + a_1x + a_2y + a_3xy$  that interpolated the boundary data at

	$\text{rerr}_2(f_1)$	$\text{rerr}_2(f_2)$	$\text{rerr}_2(f_3)$
M=2	$2.36676 \times 10^{-1}$	$2.14194 \times 10^{-2}$	$2.31158 \times 10^{-3}$
M=3	$1.00467 \times 10^{-1}$	$1.04072 \times 10^{-2}$	$1.3035 \times 10^{-3}$
M=5	$5.79567 \times 10^{-2}$	$5.45324 \times 10^{-3}$	$5.9589 \times 10^{-4}$

TABLE 9. Relative errors of the Steklov approximations of  $f_1, f_2$ , and  $f_3$ , respectively where  $h = 0.5$

the 4 corners of the rectangle. Then the Steklov approximations of solutions of Laplace's equation subject to the reduced boundary condition  $g_1(z) := g(z) - g_0(z)$  for  $z \in \partial\Omega$  were observed to be better (have smaller error) than those for the boundary data  $g$ .

Table (10) shows the comparison of relative errors of Steklov approximations of  $f_1$  and  $f_1 + 4$ . Note that  $f_1 + 4$  is the reduced boundary condition of  $f_1$  such that the value of the function at the 4 corners of  $R_1$  is zero.

	$\text{rerr}_\infty(f_1)$	$\text{rerr}_\infty(f_1 + 4)$	$\text{rerr}_2(f_1)$	$\text{rerr}_2(f_1 + 4)$
M=2	$6.59553 \times 10^{-3}$	$5.27642 \times 10^{-3}$	$5.22051 \times 10^{-3}$	$2.54632 \times 10^{-3}$
M=3	$2.28748 \times 10^{-3}$	$1.82998 \times 10^{-3}$	$1.57535 \times 10^{-3}$	$7.6838 \times 10^{-4}$
M=5	$5.55757 \times 10^{-4}$	$4.46061 \times 10^{-4}$	$3.1167 \times 10^{-4}$	$1.52018 \times 10^{-4}$

TABLE 10. Relative errors of Steklov approximations of  $f_1$  and  $f_1 + 4$  where  $h = 1$

In his thesis Cho [9] chapter 4, also investigated the approximation of harmonic functions by eigenfunctions of the Neumann Laplacian on a rectangle. Even though such eigenfunctions form an orthogonal basis of  $H^1(\Omega)$ , finite approximations involving the first M eigenfunctions were found to provide poor approximation properties for harmonic functions in  $\mathcal{H}(\Omega)$ .

## 6. APPROXIMATIONS OF SOLUTIONS OF ROBIN HARMONIC BOUNDARY VALUE PROBLEMS

When the first M harmonic Steklov eigenfunctions and eigenvalues are known, the associated Galerkin approximations of Robin or Neumann boundary value problems for Laplace's equations may be found. See Steinbach [18], chapter 8 or Zeidler [19] chapter 19 for descriptions of such constructions and their general properties. Here some specific error analyses for harmonic functions will be proved and some numerical results will be described in the next section.

A function  $u \in \mathcal{H}(\Omega)$  is said to be a (finite-energy) solution of the Robin harmonic boundary value problem on  $\Omega$  provided it satisfies

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx dy + b \int_{\partial\Omega} u v \, d\sigma = \int_{\partial\Omega} g v \, d\sigma \quad \text{for all } v \in H^1(\Omega). \quad (6.1)$$

When (B1) holds, standard variational arguments guarantee the existence and uniqueness of solutions of (6.1) in  $\mathcal{H}(\Omega)$ . The solution is denoted  $E_b g$  and satisfies the Robin boundary condition  $D_\nu u + bu = g$  on  $\partial\Omega$  in a weak sense. For  $b > 0$ , it is

$$\tilde{u}(x, y) = E_b g(x, y) := \lim_{M \rightarrow \infty} \sum_{j=0}^M \frac{\hat{g}_j \tilde{s}_j(x, y)}{b + \delta_j} \quad \text{for } (x, y) \in \Omega. \quad (6.2)$$

This limit exists in the  $H^1$ -norm provided  $g \in H^{-1/2}(\partial\Omega)$  as described in [3], section 10. In particular, this holds when  $g \in L^2(\partial\Omega, d\sigma)$ ; note that even for linear functions on a rectangle, the Robin or Neumann data  $g$  may be discontinuous on the boundary - so a useful analysis should allow such  $g$ .

When  $g_M$  is given by (3.8), take  $v = \tilde{s}_j$  in (6.1) to find that the solution is

$$u_M(x, y) := E_b g_M(x, y) = \frac{\bar{g}}{b} + \sum_{j=1}^M \frac{\hat{g}_j}{b + \tilde{\delta}_j} \tilde{s}_j(x, y) \quad \text{on } \bar{\Omega}. \quad (6.3)$$

Here  $\tilde{\delta}_j = \delta_j / |\partial\Omega|$ . That is, after the Steklov spectrum has been found, the  $M$ -th Galerkin approximation of  $E_b g$ , just requires that the Steklov coefficients  $\hat{g}_j := \langle g, \tilde{s}_j \rangle_{\partial\Omega}$  be evaluated as in (3.8).

The error estimate for these approximations is the following

**Theorem 6.1.** *Assume (B1) holds,  $b > 0$ ,  $g \in L^2(\partial\Omega, d\sigma)$  and  $g_M$  is defined by (3.8). Then the function  $u_M$  of (6.3) is in  $\mathcal{H}(\Omega)$  and*

$$\|E_b g - u_M\|_{\partial}^2 \leq \frac{1 + \delta_{M+1}}{(b + \delta_{M+1})^2} [\|g\|_{2, \partial\Omega}^2 - \|g_M\|_{2, \partial\Omega}^2] \quad (6.4)$$

Moreover the functions  $u_M$  converge uniformly to  $E_b g$  on compact subsets of  $\Omega$ .

*Proof.* From (6.1) and (6.3) one sees that

$$E_b g(x, y) - E_b g_M(x, y) = \sum_{j=M+1}^{\infty} \frac{\hat{g}_j}{b + \tilde{\delta}_j} \tilde{s}_j(x, y) \quad \text{on } \bar{\Omega}$$

Evaluating the  $\partial$ -norm of this yields, using the orthogonality of the eigenfunctions, that

$$\|E_b g - E_b g_M\|_{\partial}^2 = \sum_{j=M+1}^{\infty} \frac{\hat{g}_j^2 (1 + \delta_j)}{(b + \tilde{\delta}_j)^2}$$

Thus

$$\|E_b g - E_b g_M\|_{\partial}^2 \leq \frac{1 + \delta_{M+1}}{(b + \hat{\delta}_{M+1})^2} \|g - g_M\|_{2, \partial\Omega}^2 \quad (6.5)$$

Since  $\delta_M$  increase to infinity, the coefficient here is bounded so  $E_b g_M$  converges to  $E_b g$  in  $H^1(\Omega)$ . This equation implies (6.4) as the Steklov eigenfunctions are  $L^2$ -orthogonal on  $\partial\Omega$ . Again the uniform convergence on compact subsets of  $\Omega$  is a standard result for harmonic functions.  $\square$

The estimate in (6.4) shows again that  $H^1$ -error bounds for  $E_b g$  on  $\Omega$  may be found in terms of norms of  $g - g_M$  on  $\partial\Omega$ . Some computational results for specific examples are described in the next section.

When the Neumann boundary condition ( $b = 0$ ) holds then (6.2) holds provided  $\bar{g} = 0$  and the solution is unique up to a constant. The minimum norm solution now is

$$\tilde{u}(x, y) = E_N g(x, y) := \lim_{M \rightarrow \infty} \sum_{j=1}^M \frac{\hat{g}_j}{\delta_j} \tilde{s}_j(x, y) \quad \text{for } (x, y) \in \Omega. \quad (6.6)$$

Let  $u_M$  be this  $M$ -th partial sum, then  $u_M$  converges to  $E_N g$  in norm on  $H^1(\Omega)$  and  $E_N$  is a continuous map of  $H^{-1/2}(\partial\Omega)$  to  $\mathcal{H}(\Omega)$ . See section 10 of [3] for more details.

The following error estimate for these approximations is proved using the same arguments as those for theorem 6.1.

**Theorem 6.2.** *Assume (B1) holds,  $g \in L^2(\partial\Omega, d\sigma)$ ,  $\bar{g} = 0$  and  $g_M$  is defined by (3.8). Then  $u_M$  defined by (6.6) is in  $\mathcal{H}(\Omega)$  and*

$$\|E_N g - u_M\|_{\partial}^2 \leq \frac{1 + \delta_{M+1}}{(b + \delta_{M+1})^2} [\|g\|_{2, \partial\Omega}^2 - \|g_M\|_{2, \partial\Omega}^2] \quad (6.7)$$

Moreover the functions  $u_M$  converge uniformly to  $E_b g$  on compact subsets of  $\Omega$ .

## 7. COMPUTATION OF SOLUTIONS OF ROBIN HARMONIC BOUNDARY VALUE PROBLEMS

The results of the preceding section provide representations of the solutions of Robin and Neumann problems for the Laplacian in terms of the harmonic Steklov eigenproblems. Our observations are that approximations with relatively few (16-40) Steklov eigenfunctions compared quite well with numerical solutions obtained using finite element software such as FreeFem++ (see [16]).

Rather than compare the results with such software, however, we will present some data about comparisons with problems with exact solutions to illustrate the phenomenology observed. In particular we observed good approximations away from the boundary and some difficulty in handling discontinuity in the data  $g$  at points of discontinuity - even when the solution is nice. There is a Gibb's type effect in this case.

Denote  $\Gamma_1, \Gamma_2, \Gamma_3$ , and  $\Gamma_4$  to be the side with  $x = 1, y = h, x = -1$ , and  $y = -h$ , respectively such that  $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ .

### 7.1. Neumann harmonic boundary value problem.

Consider the boundary value problem on  $\Omega = R_h$

$$\Delta u = 0 \quad \text{on } \mathbb{R}_h \text{ with } D_\nu u = g \quad \text{on } \partial\Omega \quad (7.1)$$

with Dirichlet data

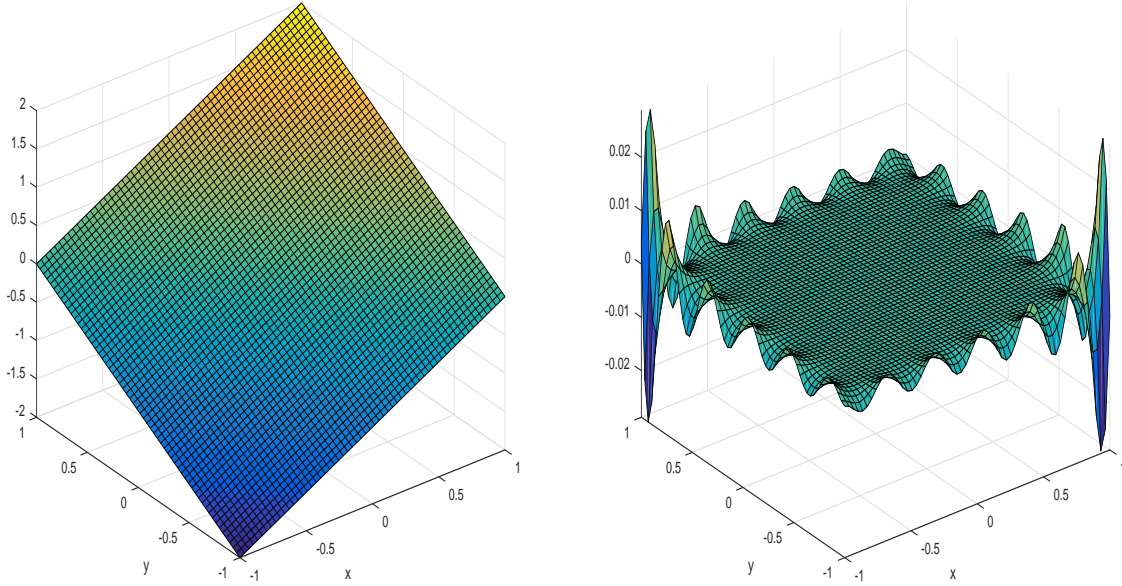
$$g(x, y) = \begin{cases} +1 & \text{on } \Gamma_1 \text{ and } \Gamma_2 \\ -1 & \text{on } \Gamma_3 \text{ and } \Gamma_4 \end{cases} \quad (7.2)$$

We note that this example has a unique solution  $u(x, y) = x + y$  with mean value zero on  $R_h$ . This solution is infinitely differentiable but the boundary data  $g$  is discontinuous at  $(-1, h)$  and  $(1, -h)$  because the domain  $R_h$  has corners.

A graph of the numerical solution and of the error  $u - u_5$  of the solution with  $M = 5$  is given in figure 2.

	$\text{rerr}_\infty(u)$	$\text{rerr}_2(u)$
M=2	$3.44988 \times 10^{-2}$	$2.17341 \times 10^{-2}$
M=3	$2.34853 \times 10^{-2}$	$1.23794 \times 10^{-2}$
M=5	$1.43896 \times 10^{-2}$	$5.98271 \times 10^{-3}$

TABLE 11. Relative error of the Steklov approximation of the solution of (7.1) with the boundary condition (7.2) where  $h = 1$



(a) Steklov approximation,  $u_5$

(b) Error in the solution,  $u - u_5$

FIGURE 2. Numerical results of the Steklov approximation of the solution of (7.1) with the boundary condition (7.2) where  $h = 1$

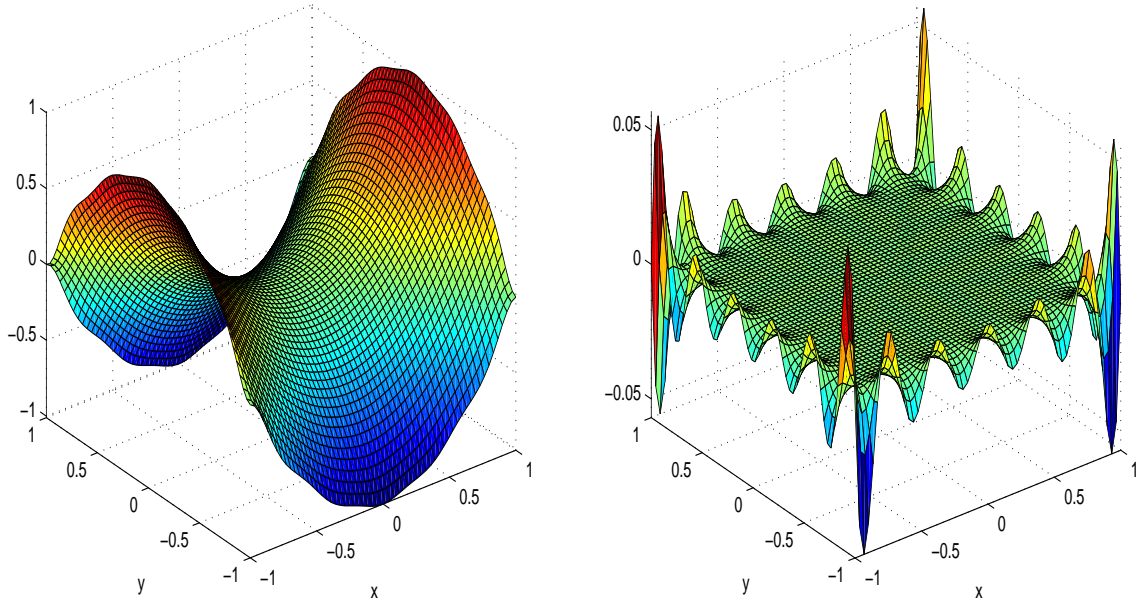
Another Neumann problem (7.1) on  $\Omega = R_h$  used  $g$  defined by

$$g(x, y) = \begin{cases} +2 & \text{on } \Gamma_1 \text{ and } \Gamma_3 \\ -2h & \text{on } \Gamma_2 \text{ and } \Gamma_4 \end{cases} \quad (7.3)$$

This problem has a unique solution  $u(x, y) = x^2 - y^2$  with mean value zero on the rectangle. This solution is a well-known saddle function but now the boundary data  $g$  is discontinuous at each corner. Graphs of the Steklov approximation with  $M = 5$  and the error  $u - u_5$  are provided in figure 3.

	$\text{rerr}_\infty(u)$	$\text{rerr}_2(u)$
M=2	$9.07987 \times 10^{-2}$	$1.32590 \times 10^{-1}$
M=3	$5.34729 \times 10^{-2}$	$9.20000 \times 10^{-2}$
M=5	$2.64002 \times 10^{-2}$	$5.70258 \times 10^{-2}$

TABLE 12. Relative error of the Steklov approximation of the solution of (7.1) with the boundary condition (7.3) where  $h = 1$



(a) Steklov approximation,  $u_5$

(b) Error in the solution,  $u - u_5$

FIGURE 3. Numerical results of the Steklov approximation of the solution of (7.1) with the boundary condition (7.3) where  $h = 1$

These simple examples show that the Steklov approximations of solutions of these problems provide quite good approximations in the interior of the region even for small

choices of  $M$ . The approximations satisfy the maximum principle, so the solutions are less accurate at, or near, the boundary.

## 7.2. Robin harmonic boundary value problem.

We consider a solution of the Robin harmonic boundary value problem with  $b = 1$  on  $R_h$ ,

$$\Delta u = 0 \quad \text{on } R_h \text{ with } D_\nu u + bu = g \quad \text{on } \partial\Omega \quad (7.4)$$

where  $g$  is given by

$$g(x, y) = \begin{cases} 2(e^1 \sin(y)) & \text{on } \Gamma_1 \\ e^x(\cos(h) + \sin(h)) & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \\ -e^x(\cos(h) + \sin(h)) & \text{on } \Gamma_4 \end{cases} \quad (7.5)$$

The unique solution of this problem is  $u(x, y) = e^x \sin(y)$ . The Steklov approximation with  $M = 5$  is shown in figure 4, together with a graph of the error function  $u - u_5$ . Again the relative error is quite reasonable and the approximations are very accurate away from the boundary.

	$\text{rerr}_\infty(u)$	$\text{rerr}_2(u)$
M=2	$1.51186 \times 10^{-2}$	$1.4854 \times 10^{-2}$
M=3	$9.64123 \times 10^{-3}$	$7.84911 \times 10^{-3}$
M=5	$5.60122 \times 10^{-3}$	$3.53263 \times 10^{-3}$

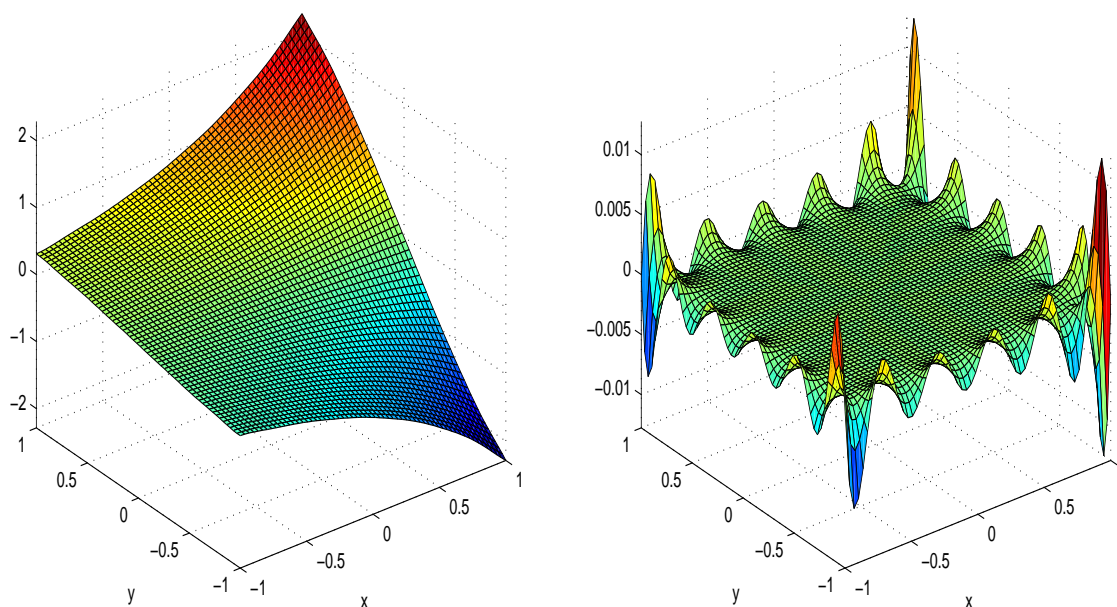
TABLE 13. Relative error of the Steklov approximation of the solution of (7.4) with the boundary condition (7.5) where  $h = 1$

These simple examples were chosen primarily to illustrate the phenomenology observed in computing Steklov approximations. There clearly are many further questions about the efficacy of such approximations but the primary observation is that low order Steklov approximations do provide good interior approximations to solutions of harmonic boundary value problems.

## REFERENCES

- [1] G. Auchmuty, "Steklov Eigenproblems and the Representation of Solutions of Elliptic Boundary Value Problems", Numerical Functional Analysis and Optimization, **25** (2004) 321-348.
- [2] G. Auchmuty, "Spectral Characterization of the Trace Spaces  $H^s(\partial\Omega)$ ", SIAM J of Mathematical Analysis, **38** (2006), 894-907.
- [3] G. Auchmuty, "Reproducing Kernels for Hilbert Spaces of Real Harmonic Functions", SIAM J Math Anal, **41**, (2009), 1994-2001.




 (a) Steklov approximation,  $u_5$ 

 (b) Error in the solution,  $u - u_5$ 

 FIGURE 4. Numerical results of the Steklov approximation of the solution of (7.4) with the boundary condition (7.5) where  $h = 1$ 

- [4] G. Auchmuty, "Bases and Comparison Results for Linear Elliptic Eigenproblems", J. Math Anal Appns, **390** (2012), 394-406.
- [5] G. Auchmuty, "The S.V.D. of the Poisson Kernel", submitted.
- [6] G. Auchmuty and M.Cho, "Boundary Integrals and Approximations of Harmonic Functions", Numerical Functional Anal. & Optimization **36** (2015), 687-703.
- [7] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd ed, Springer, N.Y. 2001.
- [8] Pan Cheng, Zhi Lin and Wenzhong Zhang, "Five-order Algorithms for solving Laplace's Steklov Eigenvalue on Polygon by Mechanical Quadrature Methods", J. Computational Analysis and Applications **16** (2015) 138-148.
- [9] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhauser, Basel (2009).
- [10] M. Cho, "Steklov Eigenproblems and Approximations of Harmonic Functions", Ph.D. thesis, University of Houston, Houston, Tx (2014)
- [11] D. Daners, "Inverse Positivity for general Robin Problems on Lipschitz Domains", Arch. Math. **92**, (2009), 57-69.
- [12] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol 1, Springer Verlag, Berlin (1988).
- [13] G. Fichera, "Su un principio di dualita per talune formole di maggiorazione relative alle equazioni differenziali", Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **19** (1955), 411-418.
- [14] A. Girouard and I. Polterovich, "Spectral Geometry of the Steklov Problem", arXiv:1411.6567v1.
- [15] P. Grisvard, *Elliptic problems in non-smooth Domains*, Pitman, Boston, (1985).
- [16] Hecht, F. New development in FreeFem++. J. Numer. Math. 20 (2012), no. 3-4, 251265. 65Y15

- [17] P.Kloucek, D.C. Sorensen and J.L. Wightman, "The Approximation and Computation of a Basis of the Trace Space  $H^{1/2}$ ", J. Scientific Computing, **32** (2007), 73-108.
- [18] O. Steinbach, *Numerical Approximation Methods for Elliptic Boundary Value Problems*, Springer, New York (2008).
- [19] E. Zeidler, *Nonlinear Functional Analysis and its Applications, IIA:* , Springer Verlag, New York (1985).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008

*E-mail address:* auchmuty@uh.edu

SCHOOL OF MATHEMATICAL SCIENCES, ROCHESTER INSTITUTE OF TECHNOLOGY, ROCHESTER,  
NEW YORK, 14623

*E-mail address:* mxcsma1@rit.edu